RELATIVISTIC CHROMOHYDRODYNAMICS AND YANG–MILLS VLASOV PLASMA

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The equations of special-relativistic, classical, ideal fluids interacting self-consistently with a Yang–Mills field are obtained through the cold plasma limit from the relativistic Yang–Mills Vlasov kinetic equations. For each system of equations the underlying hamiltonian structure is found.

Introduction. Special-relativistic fluid effects tend to be important in astrophysics. For example, the core of a neutron star is sometimes postulated to be a "quark soup", i.e., a classical fluid composed of quarks and gluons, at relativistically high temperatures. Reality, of course, can be much more complicated than this, with, for example, effects of superfluid phase transitions, particle creation, and general relativity coming into play. Here, though, we address only the problem of the description of an ideal, classical fluid, including the effects of special relativity and Yang–Mills internal degrees of freedom. Previously, chromohydrodynamics (CHD), that is, the theory of ideal fluids with gauge internal degrees of freedom interacting self-consistently with a Yang–Mills field, has been derived and given a hamiltonian description for the nonrelativistic case in refs. [1,2]. Moreover, these nonrelativistic CHD equations, together with their hamiltonian structure were obtained by taking the cold plasma limit of the corresponding equations and hamiltonian structure for a Yang–Mills Vlasov plasma (YMV) [1,2]. The YMV equations for a relativistic plasma are formulated in refs. [3,4]. In this paper, we formulate the relativistic CHD equations using the cold plasma limit of the relativistic YMV equations and find the hamiltonian structure for each of these two relativistic systems. In both instances, we observe a remarkable phenomenon which suggests what might be called *the principle of hamiltonian form-invariance under relativisation*: in passing to the relativistic descriptions, the hamiltonian structures remain unchanged, while the physical variables and the hamiltonian functionals of the corresponding relativistic systems deform regularly (in the parameter c^{-2}) from their nonrelativistic forms. This phenomenon was previously observed in ref. [5] for the Maxwell–Vlasov equations, and in ref. [6] for relativistic adiabatic fluids.

The plan of this paper is as follows. First, we cast relativistic YMV into hamiltonian form. Upon passing to the cold plasma limit and introducing thermodynamic internal energy, we derive the relativistic CHD equations and their associated hamiltonian structure.

Yang–Mills Vlasov plasma. In this section, we show that the noncanonical Poisson bracket already given in refs. [1,2] produces the correct relativistic YMV equations (in the laboratory frame) from the relativistic single-particle hamiltonian. The single-particle classical phase space for YMV in a fixed, laboratory frame consists of the particle's spatial coordinates $\boldsymbol{x}$, canonically conjugate moment components $\boldsymbol{p}$, and gauge charge g. The single-particle Poisson bracket between functions of $\boldsymbol{x}, \boldsymbol{p}$, and g is taken to be [1,2]

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$$\{J, K\} = -((\partial J/\partial \boldsymbol{p}) \cdot (\partial K/\partial \boldsymbol{x}) - (\partial J/\partial \boldsymbol{x}) \cdot (\partial K/\partial \boldsymbol{p}) + \langle g, [\partial J/\partial g, \partial K/\partial g] \rangle) . \tag{1}$$

This is the direct sum of a canonical bracket in $\boldsymbol{p}, \boldsymbol{x}$, with the Lie algebraic $\mathcal{Y}^*$-bracket. Here the charge g belongs to the dual $\mathcal{Y}^*$ of the gauge-symmetry Lie algebra $\mathcal{Y}$. The relativistic single-particle hamiltonian (routhian) is

$$H_1 = c^2[(|\boldsymbol{p} - \langle g, \boldsymbol{A}\rangle|^2/c^2 + m^2)^{1/2} - m] - \langle g, A_0 \rangle , \tag{2}$$

where $A_\mu = (A_0, \boldsymbol{A})$ is the Yang–Mills vector potential taking values in $\mathcal{Y}$, m the particle mass, and c the speed of light. Greek indices $\mu, \nu, \sigma, \alpha, \beta$ run from 0 to n (= 3), Latin indices i, j, k from 1 to n and a, b, r from 1 to $N = \dim \mathcal{Y}$; summation over repeated indices is implied; ad: $\mathcal{Y} \to \operatorname{End} \mathcal{Y}$ denotes multiplication in $\mathcal{Y}$: $\operatorname{ad}(y)z = [y, z]$; a map $\mathcal{Y} \to \mathcal{Y}^*, \mathcal{Y} \ni y \mapsto {}^*y \in \mathcal{Y}^*$ is defined by the rule $\langle y^*, z \rangle = (y, z)$ where (,) is an invariant symmetric nondegenerate form on $\mathcal{Y}$ (e.g., the Killing form, for $\mathcal{Y}$ semi-simple); t^r_{ab} are the structure constants of $\mathcal{Y}$ in a basis with elements e_b: $g = g_a e^a$, where e^a are the elements of the dual basis.

From the hamiltonian (2) one may derive

$$\ddot{x}_i = \langle g, E_i \rangle - \langle g, \dot{x}_j B_{ji} \rangle , \tag{3}$$

which is the Yang–Mills analogue of the Lorentz force; the fields E_i and B_{ij} are defined in terms of potentials $\boldsymbol{A}$ and A_0 by

$$E_i = -\partial A_i/\partial t + A_{0,i} - [A_i, A_0] , \quad B_{ij} = A_{i,j} - A_{j,i} + [A_i, A_j] . \tag{4}$$

$$\dot{g} = \operatorname{ad}(\partial H_1/\partial g)^* g = \operatorname{ad}(A_0 + \dot{\boldsymbol{x}} \cdot \boldsymbol{A})^* g \quad (\text{in components:} \quad \dot{g}_a = t^r_{ab}(A^b_0 + \dot{\boldsymbol{x}} \cdot \boldsymbol{A}^b) g_r) . \tag{5}$$

Eqs. (3) and (5) are Wong's equations [7] for the motion of a gauge-charged particle in a Yang–Mills field.

The Poisson bracket for YMV in single-particle phase space is defined for any two functionals $\mathcal{G}[f]$ and $\mathcal{K}[f]$ depending on the distribution function f on phase space to be

$$\{\mathcal{G}[f], \mathcal{K}[f]\}_f = \int f\{\delta\mathcal{G}/\delta f, \delta\mathcal{K}/\delta f\}_1 \, d^n x \, d^n p \, d^N g . \tag{6}$$

The full hamiltonian structure of relativistic YMV is *exactly the same as the nonrelativistic one* derived in refs. [1, 2], namely, the direct sum of (6) with a canonical structure for the fields ${}^*\boldsymbol{E}$ and $\boldsymbol{A}$:

$$\partial f/\partial t = \{\mathcal{H}, f\}_f (= -\{\delta\mathcal{H}/\delta f, f\}_1) , \quad \partial \boldsymbol{A}/\partial t = -\delta\mathcal{H}/\delta\, {}^*\boldsymbol{E} , \quad \partial\, {}^*\boldsymbol{E}/\partial t = \delta\mathcal{H}/\delta \boldsymbol{A} . \tag{7}$$

The hamiltonian $\mathcal{H}$ in (7) is

$$\mathcal{H} = \int H_1 f \, d^n x \, d^n p \, d^N g + \int (\langle {}^*E_i, A_{0,i} - [A_i, A_0] \rangle + \tfrac{1}{2}\langle {}^*E_i, E_i \rangle + \tfrac{1}{4}\langle {}^*B_{ij}, B_{ij} \rangle) \, d^n x , \tag{8}$$

where A_0 in this section is considered as a parameter. A proper hamiltonian description has $A_0 = 0$. Transforming eqs. (7a) to kinetic momentum $\boldsymbol{p} - \langle g, \boldsymbol{A} \rangle$ produces the relativistic YMV equations in refs. [3,4], when written in the laboratory frame.

Remark: when $\mathcal{Y}$ is abelian, this hamiltonian structure has a counterpart in the space of variables where the vector potential $\boldsymbol{A}$ is replaced by the magnetic field strength B_{ij}. The latter structure was derived in ref. [5].

As in refs. [1,2], passing to the cold plasma limit by considering only the first moments $\rho = \langle 1 \rangle$, $\boldsymbol{M} = \langle \boldsymbol{p} \rangle$, $G = \langle g \rangle$, where $\langle \cdot \rangle = \int \cdot f \, d^n p \, d^N g$, one obtains the barotropic CHD equations. Adding entropy dependence to the internal energy, we get the full CHD system, treated in the text section.

CHD. The relativistic CHD equations express conservation of energy–momentum and particle number, combined with the Yang–Mills field equations,

$$\partial_\mu T^{\mu\nu} = 0 , \quad \partial_\mu(\rho_0 u^\mu) = 0 , \quad D_\mu F^{\mu\nu} = j^\mu , \quad D_{[\sigma} \widetilde{F}^{\mu\nu]} = 0 , \tag{9}$$

where we use the following additional notation: We work in a flat space–time with coordinates x^μ, where x^0

$= ct$ is the real time coordinate. The signature of the metric tensor $g_{\alpha\beta}$ is $n-1$ $(=2)$. Derivatives are denoted $\partial_\mu = \partial/\partial x^\mu$. The energy momentum tensor is $T^{\mu\nu}$, u^μ is the particle velocity vector, ρ_0 is proper density in the frame of a moving fluid element, $F^{\mu\nu}$ is the Yang–Mills field strength tensor, $j^\nu = G_0 u^\nu$ is current density, where G_0 is the gauge charge density in the proper frame; $\widetilde{F}^{\mu\nu}$ is the tensor dual to $F^{\mu\nu}$, $\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$, D_μ is the covariant derivative, $D_\mu = \partial_\mu - \mathrm{ad}(A_\mu)$. In terms of A, $\mathrm{ad}(F_{\nu\mu}) = [D_\mu, D_\nu]$.

The expression for energy–momentum is $T^{\mu\nu} = T^{\mu\nu}_{\mathrm{MAT}} + T^{\mu\nu}_{\mathrm{YM}}$, where $T^{\mu\nu}_{\mathrm{MAT}} = \rho_0 w u^\mu u^\nu + p_0 g^{\mu\nu}$ and $T^{\mu\nu}_{\mathrm{YM}} = -\frac{1}{4}g^{\mu\nu}\langle {}^*F_{\alpha\beta}, F^{\alpha\beta}\rangle + \langle {}^*F^{\mu\alpha}, F^\nu{}_\alpha\rangle$, where $w = 1 + (e_0 + p_0/\rho_0)c^{-2}$, p_0 is pressure, and e_0 is specific internal energy, all evaluated in the proper frame of the fluid.

In the nonrelativistic limit, eqs. (9) reduce to the CHD equations given in refs. [1,2]. As a consequence of the antisymmetry of $F^{\mu\nu}$ and eq. (9c), charge is conserved: $D_\nu j^\nu = 0$.

To treat a covariant system by the hamiltonian formalism, one rewrites the covariant equations in a fixed frame of ref. [8]. For our CHD system we choose the laboratory frame (to get the eulerian description). With $u^\mu = \gamma(1, v^i/c)$, and $\gamma = (1 - v^2/c^2)^{-1/2}$, eqs. (9) become

$$\partial_t N_i = -\partial_j(N_i v^j + \delta_i^j p_0) + G(E_i + B_{ij}v^j)\,, \quad \partial_t \rho = -\partial_i(\rho v^i)\,, \quad \partial_t \eta = -\eta_{,i}v^i\,, \tag{10}$$

$$D_t({}^*E^i) = -Gv^i + {}^*(D_k B^{ki})\,, \quad D_t A_i = -E_i\,, \quad D_t^* G = -D_i^*(Gv^i)\,, \tag{11'}$$

where $D^* = \nabla + \mathrm{ad}(A)^*$, $D_t^* = \partial/\partial t + \mathrm{ad}(A_0)^*$, $B^{ki} = -F^{ki}$, N, ρ, η, and G are, respectively, the hydrodynamic momentum density, mass density, specific entropy, and gauge charge density in the laboratory frame. The velocity of the fluid relative to the laboratory frame is denoted by $\mathbf{v}$. The laboratory frame quantities are related to the proper frame quantities by $\rho = \gamma\rho_0$, $N = \gamma^2\rho_0 \mathbf{v} w$, $\eta = \eta_0$, $G = \gamma G_0$. Eqs. (10), (11') preserve Gauss' law (the zero-component of (9c)]

$$D_i^*({}^*E^i) + G = 0\,. \tag{12}$$

To make eqs. (10), (11') into a dynamical system we choose radiation gauge $A_0 = 0$. Then eqs. (11') become

$$\partial_t {}^*E^i = -Gv^i + {}^*(D_k B^{ki})\,, \quad \partial_t A_i = -E^i\,, \quad \partial_t G = -D_i^*(Gv^i)\,. \tag{11}$$

The main result of this section is that the relativistic equations (10), (11) are hamiltonian with the following Poisson bracket

$$\{I, J\} = -\int \mathrm{d}^n x \left\{ \left(\frac{\delta J}{\delta\rho}\partial_i\rho + \frac{\delta J}{\delta\eta}\eta_{,i} + \frac{\delta J}{\delta N_j}(N_i\partial_j + \partial_i N_j - \langle G, B^{ij}\rangle) - \left\langle G, \frac{\delta J}{\delta\, {}^*E^i}\right\rangle\right)\frac{\delta I}{\delta N_i} + \left\langle D_i^* G\frac{\delta I}{\delta N_i}, \frac{\delta J}{\delta G}\right\rangle \right.$$
$$\left. + \left\langle G, \left[\frac{\delta I}{\delta G}, \frac{\delta J}{\delta G}\right]\right\rangle + \frac{\delta J}{\delta N_j}\left(\left\langle G, D_j\frac{\delta I}{\delta G}\right\rangle + \rho\partial_j\frac{\delta I}{\delta\rho} - \eta_{,j}\frac{\delta I}{\delta\eta} + \left\langle G, \frac{\delta I}{\delta\, {}^*E^j}\right\rangle\right) + \frac{\delta J}{\delta A_i}\frac{\delta I}{\delta\, {}^*E^i} - \frac{\delta I}{\delta A_i}\right\}, \tag{13}$$

and hamiltonian

$$H = \int T^{00}\,\mathrm{d}^n x = \int \mathrm{d}^n x\ \{c^2[(|N|^2/c^2 + (\rho w)^2)^{1/2} - \rho] - p_0 + \tfrac{1}{2}\langle {}^*E_i, E^i\rangle + \tfrac{1}{4}\langle {}^*B_{ij}, B^{ij}\rangle\ . \tag{14}$$

This result follows immediately by substituting the identities

$$\delta H/\delta N = \mathbf{v}\,, \quad \delta H/\delta\rho = c^2(w\gamma^{-1} - 1)\,, \quad \delta H/\delta\eta = c^2\rho_0 w_{,\eta} - p_{,\eta}\,,$$
$$\delta H/\delta\, {}^*E_i = E^i\,, \quad \delta H/\delta A_i = {}^*(D_k B^{ki})\,, \quad \delta H/\delta G = 0\,,$$

into the equations $F_{,t} = \{H, F\}$ for $F \in \{N, \rho, \eta, G, {}^*E, A\}$ and using (13). The Poisson bracket (13) satisfies the Jacobi identity because this bracket is identical to the bracket for nonrelativistic CHD derived in refs. [1,2]; except instead of the total momentum density M in refs. [1,2] we use here the hydrodynamic momentum density $N =$

$M - \langle G, A \rangle$. Notice that when $c^{-2} \to 0$, the motion equations (10), (11), the hamiltonian (14), and the functional derivatives (15) all tend to their nonrelativistic counterparts. Thus, relativistic CHD is a regular, structure preserving deformation (with parameter c^{-2}) of nonrelativistic CHD.

For multiple gauge charge species (flavors) all variables except $^*E$ and A acquire species labels, and one sums over species in Gauss' law (12), in the Poisson bracket (13), and in the hamiltonian (14), The resulting relativistic multi-species CHD system has a number of limiting cases. In the abelian, single-species case, eqs. (9) reduce to the equations for a classical relativistic charged fluid (ref. [9], ch. 3). In the abelian nonrelativistic case, one recovers the hamiltonian structure for the multi fluid plasma system [10,11] which was found earlier [12] for the single-species pressureless relativistic fluid. Notice that in the abelian case, the multi-species version of the Poisson bracket (13) written in terms of the vector potential A, has a counterpart in terms of the field strength B^{ij}. This correlation was observed in the nonrelativistic case in ref. [11]. Because the relativistic and nonrelativistic A-brackets are the same, this correlation persists in the relativistic case, and since the hamiltonian depends upon A only through B^{ij}, the resulting hamiltonian description is gauge-invariant.

As we saw above, the preservation of hamiltonian structure when passing from nonrelativistic to relativistic description occurs for CHD and YMV. This was also observed for the case of relativistic fluids in ref. [6] and for the Maxwell–Vlasov system in ref. [5], where the relativistic Poisson bracket (in the B^{ij}-space) was found to be of the same form as the nonrelativistic one [13]. The observation that various hamiltonian theories have the same Poisson bracket in either nonrelativistic or relativistic variables in a fixed frame, strongly indicates that there exists an underlying principle of form-invariance upon relativisation of the hamiltonian structures of fluids and plasmas.

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References

[1] J. Gibbons, D.D. Holm and B.A. Kupershmidt, Phys. Lett. 90A (1982) 281.
[2] J. Gibbons, D.D. Holm and B.A. Kupershmidt, Physica 6D (1983) 179.
[3] U. Heinz, in: Quark matter formation and heavy ion collisions, eds. M. Jacob and H. Satz (World Scientific, Singapore, 1982) p. 439.
[4] U. Heinz, Phys. Rev. Lett. 51 (1983) 351.
[5] I. Bialynicki-Birula and J.C. Hubbard, Physica 128A (1984), to be published.
[6] D.D. Holm and B.A. Kupershmidt, Phys. Lett. 101A (1984) 23.
[7] S.K. Wong, Nuovo Cimento 65A (1970) 689.
[8] P.A.M. Dirac, in: Recent developments in general relativity, A collection of papers dedicated to L. Infeld (Pergamon, New York, 1962).
[9] A. Lichnerowitz, Relativistic hydrodynamics and magnetohydrodynamics (Benjamin, New York, 1967).
[10] R.G. Spencer and A.N. Kaufman, Phys. Rev. A25 (1982) 2437.
[11] D.D. Holm and B.A. Kupershmidt, Physica 6D (1983) 347.
[12] I. Bialynicki-Birula and Z. Iwinski, Rep. Math. Phys. 4 (1973) 139.
[13] J.E. Marsden and A. Weinstein, Physica 4D (1982) 394.